Pergamon

Int. J. Solids Structures Vol. 35, Nos 26–27, pp. 3595–3610, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(98)00024–9

EXACT HIGHER ORDER SOLUTIONS FOR A SIMPLE ADAPTIVE STRUCTURE

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(*Received* 2 *August* 1997; *in revised form* 23 *December* 1997)

Abstract—The analysis of an adaptive beam composed by a passive layer and two surface bonded induced strain actuators is considered. A simple higher order beam model is formulated and exact solutions are obtained for a case of membrane actuation and for a pure bending case. The obtained solutions are then discussed in terms of the main geometrical parameters of the system and compared with the classical closed form solutions based on Euler–Bernoulli models. As a result the interaction mechanism between the passive and the active part of the structure is better described than in existing closed form models. Moreover the present model allows the description of the edge effect which occurs close to the free boundary of the considered structure.

1. INTRODUCTION

The analysis of intelligent structures composed by passive materials and active elements has recently received an increasing amount of interest (Crawley and de Louis, 1987; Crawley, 1994). Active materials such as piezoceramics (PZT) and piezopolymers (PVDF) are being studied as parts of vibration reduction mechanisms. Their role as sensors and actuators has been effectively demonstrated for simple feed-back control techniques both from an experimental and a theoretical point of view (Denoyer and Kwak, 1996; Gaudenzi *et al.*, 1997).

An increasing number of modeling studies has recently proposed the analysis of homogeneous piezoelectric media. Gaudenzi and Bathe (1995) proposed a general finite element procedure for the nonlinear analysis of electroelastic materials. A closed form solution for rectangular plates was recently presented by Bisegna and Maceri (1996).

Other studies consider the analysis of composite structures in which only a part of the system is made of active (piezoelectric) elements. In this case different constitutive relations are present in different regions such as in the finite element studies by Kim *et al.* (1997) and by Gaudenzi (1997).

A number of contributions focus on the analysis of layered structures for which, starting from the work of Crawley and de Louis (1987), beam and plate models of increasing degree of complexity has been presented such as in the finite element works of Saravanos *et al.* (Saravanos and Heyliger, 1995; Saravanos *et al.*, 1997) and in the case of closed form solution proposed by Ray *et al.* (1993).

In parallel with numerous existing studies on composite laminated structures (Reddy, 1987; Gaudenzi *et al.*, 1995) higher order theories for active composite laminates can be proposed for improving the modeling capabilities of classical plate theories. In these studies, such as the ones by Saravanos and Hegyliger (1995), Saravanos *et al.* (1997) and by Barboni *et al.* (1995), the hypothesis of plane section is removed and more realistic kinematic models are proposed.

In the present study exact solutions are presented for a higher order modeling approach of the behavior of an active cantilever beam. Two symmetrically surface bonded piezoelectric layers are present on the top and the bottom of a beam. A simple higher order kinematic is proposed which allows one to consider curved deformed configurations of the plane sections of the beam and more accurate simulation of the stress distributions.

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The results, obtained both for a bending and a membrane case, are compared with the ones obtained for the case of plane sections which remain normal to the axis of the beam after deformation (Euler–Bernoulli case). The effects of the geometry and the material parameter on the proposed solutions are also shown and discussed in detail.

2. THEORETICAL FORMULATION

The considered structure consists of two actuating piezoelectric layers perfectly bonded to a passive cantilever beam structure (Fig. 1). The linear response of the system is studied in the bidimensional case. The structure is symmetric with respect to its middle plane and all its parts are governed by an isotropic elastic behavior.

The piezoelectric parts exhibit a coupled piezo-elastic constitutive behavior in which the electric field is supposed to be given for each point of the material. Namely a constant electric field is assumed. For this reason the actuation mechanism is represented in the model simply by applied actuation strain components.

In the following subsections the mathematical formulation of the problem is given and the governing equations in terms of generalized unknown displacement are obtained.

2.1. *Kinematics*

For the beam of Fig. 1 consisting of two actuating layers of thickness t_a and a passive structure of thickness t_s the following form of displacement is assumed:

$$u(x,z) = u_0(x)+u_1(x)z+u_2(x)z^2+u_3(x)z^3$$
$$w(x,z) = w_0(x)+w_1(x)z+w_2(x)z^2 \tag{1}$$

which can be considered as a power series expansion of the displacement components along the z-axis confined to a limited number of terms.

In this way a segment normal to the middle plane of the structure may be assumed after deformation of a curved shape. As a consequence the expression of total strain components assumes the form, where $[\cdot]'$ indicates $\mathrm{d}[\cdot]/\mathrm{d}x$:

$$\varepsilon_x = u_0'(x)+u_1'(x)z+u_2'(x)z^2+u_3'(x)z^3$$
$$\varepsilon_z = w_1(x)+2w_2(x)z$$

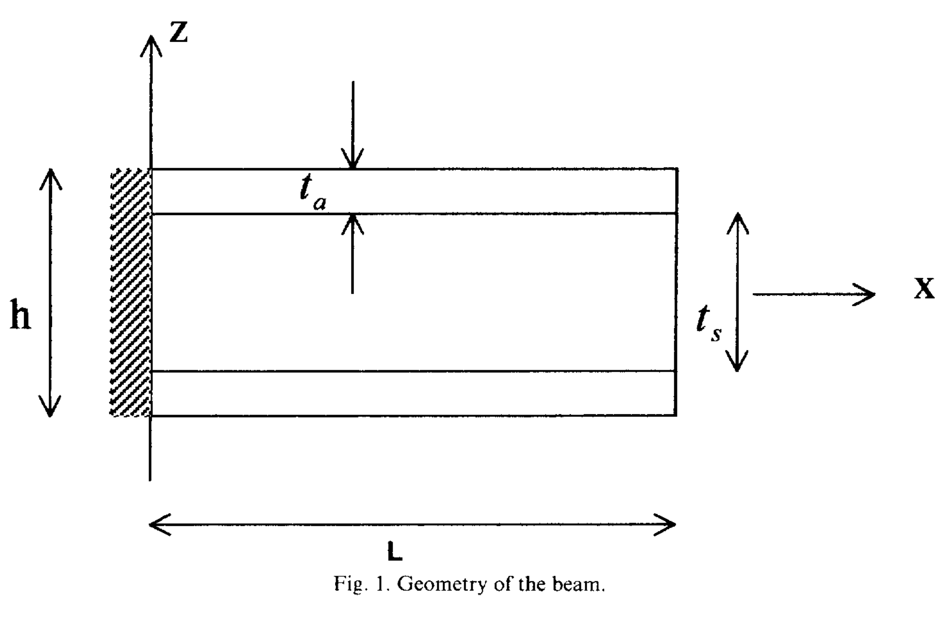

Fig. 1. Geometry of the beam.

$$\gamma_{xz} = u_1(x) + w_0'(x) + [2u_2(x) + w_1'(x)]z + [3u_3(x) + w_2'(x)]z^2 \tag{2}$$

2.2. *Constitutive relations*

Passive plate structure:

$$\sigma_x = \frac{E_s}{1-v_s^2}(\varepsilon_x + v_s\varepsilon_z)$$

$$\sigma_z = \frac{E_s}{1-v_s^2}(v_s\varepsilon_x + \varepsilon_z)$$

$$\tau_{xz} = G\gamma_{xz} \tag{3}$$

Active piezoelectric layers:

$$\sigma_x = \frac{E_a}{1-v_a^2}[(\varepsilon_x - \Lambda_1) + v_a(\varepsilon_z - \Lambda_3)]$$

$$\sigma_z = \frac{E_a}{1-v_a^2}[v_a(\varepsilon_x - \Lambda_1) + (\varepsilon_z - \Lambda_3)]$$

$$\tau_{xy} = G\gamma_{xy} \tag{4}$$

Λ_1, Λ_3 are the normal actuation strain components in x and z direction, respectively. They can be considered as due to an electric field applied in the z direction that is the direction of polarization of the piezoelectric material. The pedix "s" denotes the passive structure while "a" identifies the actuators.

2.3. *Governing equations*

The principle of virtual work is used to obtain the governing equations of the problem. In the absence of external applied loads the virtual internal load is equal to zero:

$$1 \cdot \int_0^l \int_{-(h/2)}^{h/2} (\sigma_x\delta\varepsilon_x + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy})\,\mathrm{d}z\,\mathrm{d}x = 0 \tag{5}$$

where a unit width of the beam has been considered.

By applying a virtual displacement δu_0 the only virtual deformation is $\delta\varepsilon_x = \delta u_0'$ and the preceding integral becomes:

$$\int_0^l \int_{-(h/2)}^{h/2} \sigma_x \delta u_0' \,\mathrm{d}z\,\mathrm{d}x = 0 \tag{6}$$

By integrating by parts, the virtual displacement δu_0 appears instead of its derivative with respect to x.

$$\int_{-(h/2)}^{h/2} \sigma_x \,\mathrm{d}z\delta u_0|_0^l - \int_0^l \int_{-(h/2)}^{h/2} \sigma_x'\delta u_0 \,\mathrm{d}x\,\mathrm{d}z = 0 \tag{7}$$

The first term generates the natural boundary conditions. In the considered case since the first term has to vanish for every possible virtual displacement, either the normal force $N_x = \int_{-(h/2)}^{h/2} \sigma_x \cdot 1 \cdot \mathrm{d}z$ or the axial displacement u_0 are forced to be zero, which yields to the condition

$$\int_0^l \int_{-(h/2)}^{h/2} \sigma'_x \delta u_0 \, dz \, dx = 0 \tag{8}$$

That is:

$$\int_0^l \left[\int_{-(t_s/2)}^{t_s/2} \frac{E_s}{1-v_s^2}(\varepsilon_x + v_s \varepsilon_z) \, dz + \int_{-(h/2)}^{-(t_s/2)} \frac{E_a}{1-v_a^2}[(\varepsilon_x - \Lambda_1) + v_a(\varepsilon_z - \Lambda_3)] \, dz \right.$$
$$\left. + \int_{t_s/2}^{h/2} \frac{E_a}{1-v_a^2}[(\varepsilon_x - \Lambda_1) + v_a(\varepsilon_z - \Lambda_3)] \, dz \right]' dx = 0 \tag{9}$$

By integrating with respect to dz, the following differential equation is derived:

$$A_1 u_0'' + A_3 u_2'' + B_1 w_1' = 0 \tag{10}$$

With the corresponding boundary conditions, respectively, obtained for $x = 0$, $x = l$:

$$u_0 = 0 \quad A_1 u_0' + B_1 w_1 + A_3 u_2' = 0 \tag{11}$$

A_1, A_3, B_1 are constant terms better defined in the following. In the same way, by taking virtual variation of the other six components of the displacement, the following set of ordinary differential equations can be obtained:

$$\begin{aligned} A_1 u_0'' + A_3 u_2'' + B_1 w_1' &= 0 \\ A_3 u_0'' - 4C_3 u_2 + A_5 u_2'' + (B_3 - 2C_3) w_1' &= 0 \\ B_1 u_0' + (B_3 - 2C_3) u_2' + A_1 w_1 - C_3 w_1'' &= L_1 \Lambda_{3m} \end{aligned} \tag{12}$$

$$\begin{aligned} C_1 u_1 - A_3 u_1'' + 3C_3 u_3 - A_5 u_3'' + C_1 w_0' + (C_3 - 2B_3) w_2' &= 0 \\ 3C_3 u_1 - A_5 u_1'' + 9C_5 u_3 - A_7 u_3'' + 3C_3 w_0' + (3C_5 - 2B_5) w_2' &= 0 \\ C_1 u_1' + 3C_3 u_3' + C_1 w_0'' + C_3 w_2'' &= 0 \\ (C_3 - 2B_3) u_1' + (3C_5 - 2B_5) u_3' + C_3 w_0'' - 4A_3 w_2 + C_5 w_2'' &= 2L_2 \Lambda_{3b} \end{aligned} \tag{13}$$

With the boundary conditions:

$$\begin{aligned} &x = 0 && x = l \\ &u_0 = 0 && A_1 u_0' + B_1 w_1 + A_3 u_2' - L_1 \Lambda_{1m} = 0 \\ &u_2 = 0 && A_3 u_0' + B_3 w_1 + A_5 u_2' - L_3 \Lambda_{1m} = 0 \\ &w_1 = 0 && 2C_3 u_2 + C_3 w_1' = 0 \\ &u_1 = 0 && A_3 u_1' + 2B_3 w_2 + A_5 u - L_2 \Lambda_{1b} = 0 \\ &u_3 = 0 && A_5 u_1' + 2B_5 w_2 + A_7 u_3' - L_4 \Lambda_{1b} = 0 \\ &w_0 = 0 && C_1 u_1 + C_1 w_0' + 3C_3 u_3 + C_3 w_2' = 0 \\ &w_2 = 0 && C_3 u_1 + C_3 w_0' + 3C_5 u_3 + C_5 w_2' = 0 \end{aligned} \tag{14}$$

Where the coefficients A_k, B_k, C_k, L_k, and the actuating terms are:

$$A_k = \frac{2}{k}\left\{\frac{E_s}{1-\nu_s^2}\left(\frac{t_s}{2}\right)^k + \frac{E_a}{1-\nu_a^2}\left[\left(\frac{h}{2}\right)^k - \left(\frac{t_s}{2}\right)^k\right]\right\}$$

$$B_k = \frac{2}{k}\left\{\frac{E_s\nu_s}{1-\nu_s^2}\left(\frac{t_s}{2}\right)^k + \frac{E_a\nu_a}{1-\nu_a^2}\left[\left(\frac{h}{2}\right)^k - \left(\frac{t_s}{2}\right)^k\right]\right\}$$

$$C_k = \frac{2}{k}\left\{G_s\left(\frac{t_s}{2}\right)^k + G_a\left[\left(\frac{h}{2}\right)^k - \left(\frac{t_s}{2}\right)^k\right]\right\}$$

$$L_k = \frac{2}{k}\frac{E_a}{1-\nu_a^2}\left[\left(\frac{h}{2}\right)^k - \left(\frac{t_s}{2}\right)^k\right] \quad h = 2t_a + t_s \tag{15}$$

$$\Lambda_{1m} = \frac{(\Lambda_1^u + \nu_a\Lambda_3^u) + (\Lambda_1^l + \nu_a\Lambda_3^l)}{2}$$

$$\Lambda_{3m} = \frac{(\Lambda_3^u + \nu_a\Lambda_1^u) + (\Lambda_3^l + \nu_a\Lambda_1^l)}{2}$$

$$\Lambda_{1b} = (\Lambda_1^u + \nu\Lambda_3^u) - \Lambda_{1m}$$

$$\Lambda_{3b} = (\Lambda_3^u + \nu_a\Lambda_1^u) - \Lambda_{3m} \tag{16}$$

The superscripts u and l stand respectively for upper or lower actuator. As it can easily be seen, these equations can be grouped in two sets: the first one (first three equations) is relevant to a membrane mode, the second one to a bending mode. Note that, since the dependence of the unknowns upon z has been established *a priori*, only total derivatives with respect to x appear in the formulation.

3. A CLOSED FORM SOLUTION FOR THE MEMBRANE MODE

(a) Let us now consider the membrane mode and restrict the analysis in the case for which $w_1(x) = 0$. The membrane case is then governed by the following set of equations

$$\begin{aligned} A_1u_0'' + A_3u_2'' &= 0 \\ A_3u_0'' - 4C_3u_2 + A_5u_2'' &= 0 \end{aligned} \tag{17}$$

With the boundary conditions:

$$x = 0 \quad \begin{cases} u_0 = 0 \\ u_2 = 0 \end{cases}$$

$$x = l \quad \begin{cases} A_1u_0' + A_3u_2' = L_1\Lambda_{1m} \\ A_3u_0' + A_5u_2' = L_3\Lambda_{1m} \end{cases} \tag{18}$$

The following solution form is assumed:

$$U(x) = \begin{Bmatrix} u_0(x) \\ u_2(x) \end{Bmatrix} = \begin{Bmatrix} u_0 \\ u_2 \end{Bmatrix} e^{\phi x} \tag{19}$$

from which the preceding system becomes:

$$A_1\phi^2 u_0 + A_3\phi^2 u_2 = 0$$
$$A_3\phi^2 u_0 - 4C_3 u_2 + A_5\phi^2 u_2 = 0 \tag{20}$$

To obtain a non trivial solution, the characteristic polynomial of this system has to vanish

$$\phi^2[(A_1A_5 - A_3^2)\phi^2 - 4C_3A_1] = 0 \tag{21}$$

For which equations the solutions are:

$$\phi_{1,2} = 0$$
$$\phi_{3,4} = \pm 2\sqrt{\frac{C_3A_1}{A_1A_5 - A_3^2}} = \pm\alpha \tag{22}$$

The solution of the system is therefore of the form:

$$U(x) = \begin{Bmatrix} u_0(x) \\ u_2(x) \end{Bmatrix} = U_1 + U_2 x + U_3 e^{-\alpha x} + U_4 e^{\alpha x} \tag{23}$$

This solution must meet the equilibrium equations, for every value of x, and the boundary conditions.

$$A_1(\alpha^2 u_{03} e^{-\alpha x} + \alpha^2 u_{04} e^{\alpha x}) + A_3(\alpha^2 u_{23} e^{-\alpha x} + \alpha^2 u_{24} e^{\alpha x}) = 0$$
$$A_3(\alpha^2 u_{03} e^{-\alpha x} + \alpha^2 u_{04} e^{\alpha x}) - 4C_3(u_{21} + u_{22}x + u_{23} e^{-\alpha x} + u_{24} e^{\alpha x})$$
$$+ A_5(\alpha^2 u_{23} e^{-\alpha x} + \alpha^2 u_{24} e^{\alpha x}) = 0$$

By imposing that the sum of the coefficients which appear multiplied by the same function of x vanishes, we obtain:

$$A_1 u_{03} + A_3 u_{23} = 0 \quad A_1 u_{04} + A_3 u_{24} = 0 \quad u_{21} = 0 \quad u_{22} = 0 \tag{24}$$

and the conditions in $x = 0$, $x = l$:

$$x = 0 \quad \begin{cases} u_{01} + u_{03} + u_{04} = 0 \\ u_{21} + u_{23} + u_{24} = 0 \end{cases}$$
$$x = l \quad \begin{cases} A_1(u_{02} - \alpha u_{03} e^{-\alpha l} + \alpha u_{04} e^{\alpha l}) + A_3(u_{22} - \alpha u_{23} e^{-\alpha l} + \alpha u_{24} e^{\alpha l}) = L_1\Lambda_{1m} \\ A_3(u_{02} - \alpha u_{03} e^{-\alpha l} + \alpha u_{04} e^{\alpha l}) + A_5(u_{22} - \alpha u_{23} e^{-\alpha l} + \alpha u_{24} e^{\alpha l}) = L_3\Lambda_{1m} \end{cases} \tag{25}$$

Finally the solution can be written as:

$$u_{01} = 0$$
$$u_{02} = \frac{L_1}{A_1}\Lambda_{1m}$$
$$u_{03} = -u_{04} = -\frac{A_3}{A_1}u_{23}$$
$$u_{21} = u_{22} = 0$$
$$u_{23} = -u_{24} = \left(L_3 - \frac{A_3}{A_1}L_1\right)\Lambda_{1m}\frac{A_1}{\alpha(A_3^2 - A_5A_1)}\frac{1}{e^{-\alpha l} + e^{\alpha l}} \tag{26}$$

$$u(x,z) = \frac{L_1}{A_1}\Lambda_{1m}x + \frac{\alpha}{4C_3A_1}\frac{e^{\alpha x}-e^{-\alpha x}}{e^{-\alpha l}+e^{\alpha l}}\left(L_3 - L_1\frac{A_3}{A_1}\right)\Lambda_{1m}(-A_3+A_1z^2) \tag{27}$$

(b) We now obtain the solution for the simple case for which the expression of $u(x,z)$ is limited to the constant term with respect to z:

$$u(x) = u_0(x) \tag{28}$$

The governing equation and the boundary conditions are in this case:

$$A_1u_0'' = 0x = 0 \quad u_0 = 0 \quad x = l \quad A_1u_0' = L_1\Lambda_{1m} \tag{29}$$

In conclusion the following solution is found:

$$u(x) = \frac{L_1}{A_1}\Lambda_{1m} \tag{30}$$

Which can be recognized as the first term obtained in the higher order case.

4. A CLOSED FORM SOLUTION FOR THE BENDING MODE

(a) We consider now the bending actuation case and restrict the analysis to the case for which the expression of $w(x,z)$ is limited to the constant term with respect to z, that is:

$$w(x) = w_0(x) \tag{31}$$

In this case the governing equations read:

$$\begin{aligned} &C_1u_1 - A_3u_1'' + 3C_3u_3 - A_5u_3'' + C_1w_0' = 0 \\ &3C_3u_1 - A_5u_1'' + 9C_5u_3 - A_7u_3'' + 3C_3w_0' = 0 \\ &C_1u_1' + 3C_3u_3' + C_1w_0'' = 0 \end{aligned} \tag{32}$$

The assumed boundary conditions are:

$$x = 0 \quad \begin{cases} u_1 = 0 \\ u_3 = 0 \\ w_0 = 0 \end{cases}$$

$$x = l \quad \begin{cases} A_3u_1' + A_5u_3' - L_2\Lambda_{1b} = 0 \\ A_5u_1' + A_7u_3' - L_4\Lambda_{1b} = 0 \\ C_1u_1 + C_1w_0' + 3C_3u_3 = 0 \end{cases} \tag{33}$$

The following solution form is assumed:

$$W(x) = \begin{Bmatrix} u_1(x) \\ u_3(x) \\ w_0(x) \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_3 \\ w_0 \end{Bmatrix} e^{\phi x} \tag{34}$$

from which the following homogeneous algebraic system is found:

$$
\begin{aligned}
&(C_1-A_3\phi^2)u_1+(3C_3-A_5\phi^2)u_3+C_1\phi w_0=0\\
&(3C_3-A_5\phi^2)u_1+(9C_5-A_7\phi^2)u_3+3C_3\phi w_0=0\\
&C_1\phi u_1+3C_3\phi u_3+C_1\phi^2 w_0=0
\end{aligned}
\tag{35}
$$

By imposing the characteristics equation to vanish:

$$
\phi^4[C_1(A_3A_7-A_5^2)\phi^2-9A_3(C_1C_5-C_3^2)]=0 \tag{36}
$$

we obtain for ϕ the following values:

$$
\begin{aligned}
&\phi_i=0 \quad i=1,2,3,4\\
&\phi_{1,2}=\pm\alpha=\pm 3\sqrt{\frac{A_3(C_1C_5-C_3^2)}{C_1(A_3A_7-A_5^2)}}
\end{aligned}
\tag{37}
$$

The solution of the system is therefore of the form:

$$
W(x)=\begin{Bmatrix} u_1(x)\\ u_3(x)\\ w_0(x)\end{Bmatrix}=W_1+W_2x+W_3x^2+W_4x^3+W_5e^{-\alpha x}+W_6e^{\alpha x} \tag{38}
$$

By substituting this expression in the governing equations we obtain:

$$
\begin{aligned}
&C_1(u_{11}+u_{12}x+u_{13}x^2+u_{14}x^3+u_{15}e^{-\alpha x}+u_{16}e^{\alpha x})-A_3(2u_{13}+6u_{14}x+\alpha^2u_{15}e^{-\alpha x}+\alpha^2u_{16}e^{\alpha x})\\
&+3C_3(u_{31}+u_{32}x+u_{33}x^2+u_{34}x^3+u_{35}e^{-\alpha x}+u_{36}e^{\alpha x})-A_5(2u_{33}+6u_{34}x+\alpha^2u_{35}e^{-\alpha x}+\alpha^2u_{36}e^{\alpha x})\\
&\qquad+C_1(w_{02}+2w_{03}x+3w_{04}x^2-\alpha w_{05}e^{-\alpha x}+\alpha w_{06}e^{\alpha x})=0\\
&3C_3(u_{11}+u_{12}+u_{13}x^2+u_{14}x^3+u_{15}e^{-\alpha x}+u_{16}e^{\alpha x})-A_5(2u_{13}+6u_{14}x+\alpha^2u_{15}e^{-\alpha x}+\alpha^2u_{16}e^{\alpha x})\\
&+9C_5(u_{31}+u_{32}x+u_{33}x^2+u_{34}x^3+u_{35}e^{-\alpha x}+u_{36}e^{\alpha x})-A_7(2u_{33}+6u_{34}x+\alpha^2u_{35}e^{-\alpha x}+\alpha^2u_{36}e^{\alpha x})\\
&\qquad\times 3C_{31}(w_{02}+2w_{03}x+3w_{04}x^2-\alpha w_{05}e^{-\alpha x}+\alpha w_{06}e^{\alpha x})=0\\
&C_1(u_{12}+2u_{13}x+3u_{14}x^2-\alpha u_{15}e^{-\alpha x}+\alpha u_{16}e^{\alpha x})+3C_3(u_{32}+2u_{33}x+3u_{34}x^2-\alpha u_{35}e^{-\alpha x}\\
&\qquad+\alpha u_{36}e^{\alpha x})+C_1(2w_{03}+6w_{04}x+\alpha^2w_{05}e^{-\alpha x}+\alpha^2w_{06}e^{\alpha x})=0
\end{aligned}
\tag{39}
$$

These equations must be satisfied for every value of x in the range $0 \leqslant x \leqslant l$, that is the sum of the coefficients which appear multiplied by the same function of x must vanish. In this way a system of twelve independent equations is found that, with the corresponding boundary conditions, can be solved to find the eighteen constant of the problem.

$$
u(x,z)=\frac{L_2}{A_3}\Lambda_{1b}xz+\frac{C_1\alpha}{9A_3(C_1C_5-C_3^2)}\frac{e^{\alpha x}-e^{-\alpha x}}{e^{\alpha l}+e^{-\alpha x}}\left(L_4-L_2\frac{A_5}{A_3}\right)\Lambda_{1b}(-A_5z+A_3z^3) \tag{40}
$$

$$
w(x,z)=-\frac{L_2}{2A_3}\Lambda_{1b}x^2+\frac{C_1A_5-3C_3A_3}{9A_3(C_1C_5-C_3^2)}\frac{e^{\alpha x}+e^{-\alpha x}-2}{e^{\alpha l}+e^{\alpha l}}\left(L_4-L_2\frac{A_5}{A_3}\right)\Lambda_{1b} \tag{41}
$$

And the curvature of the axis $k = -w''$ is:

$$w''(x) = -\frac{L_2}{A_3\Lambda_{1b}} + \frac{C_1A_5-3C_3A_3}{C_1(A_3A_7-A_5^2)}\frac{e^{\alpha x}+e^{-\alpha x}}{e^{\alpha l}+e^{-\alpha l}}\left(L_4-L_2\frac{A_5}{A_3}\right)\Lambda_b \tag{42}$$

(b) We now consider the problem in the case for which the expression of the axial displacement is limited to the term $u_1(x)$:

$$u(x, z) = u_1(x)z \tag{43}$$

therefore the governing equations read:

$$C_1u_1 - A_3u_1'' + C_1w_0' = 0 \quad C_1u_1' + C_1w_0'' = 0 \tag{44}$$

With the boundary conditions:

$$x = 0 \quad \begin{cases} u_1 = 0 \\ w_0 = 0 \end{cases}$$

$$x = l \quad \begin{cases} A_3u_1' - L_2\Lambda_{1b} = 0 \\ C_1u_1 + C_1w_0 = 0 \end{cases} \tag{45}$$

Assuming again the following form for the solution:

$$W(x) = \begin{Bmatrix} u_1(x) \\ w_0(x) \end{Bmatrix} = \begin{Bmatrix} u_1 \\ w_0 \end{Bmatrix} e^{\phi x} \tag{46}$$

and substituting it in the governing equations:

$$C_1u_1\,e^{\phi x} - \phi^2A_3u_1\,e^{\phi x} + \phi C_1w_0\,e^{\phi x} = 0 \quad C_1\phi u_1\,e^{\phi x} + C_1\phi^2w_0\,e^{\phi x} = 0 \tag{47}$$

The characteristic equation of the system is:

$$\phi^4 = 0 \tag{48}$$

Then the solution can be written as:

$$W(x) = \begin{Bmatrix} u_1(x) \\ w_0(x) \end{Bmatrix} = W_1 + W_2x + W_3x^2 + W_4x^3 \tag{49}$$

and the equilibrium equations become:

$$C_1(u_{11}+u_{12}x+u_{13}x^2+u_{14}x^3) - A_3(2u_{13}+6u_{14}x) + C_1(w_{02}+2w_{03}x+3w_{04}x^2) = 0$$
$$C_1(u_{12}+2u_{13}x+3u_{14}x^2) + C_1(2w_{03}+6w_{04}x) = 0 \tag{50}$$

These equations must be verified for every value of x in the range $0 \leqslant x \leqslant l$

$$C_1u_{11} - 2A_3u_{13} + C_1w_{02} = 0 \quad C_1u_{12} - 6A_3u_{14} + 2C_1w_{03} = 0 \quad C_1u_{13} + 3C_1w_{04} = 0 \quad u_{14} = 0 \tag{51}$$

While the boundary conditions read:

$$x = 0 \quad \begin{cases} u_{11} = 0 \\ w_{01} = 0 \end{cases}$$

$$x = l \quad \begin{cases} A_3(u_{12}+2u_{13}l+3u_{14}l^2) = L_2\Lambda_{1b} \\ C_1(u_{11}+u_{12}l+u_{13}l^2+u_{14}l^3)+C_1(w_{02}+2w_{03}l+3w_{04}l^2) = 0 \end{cases} \tag{52}$$

Finally the eight constants of the problem are obtained:

$$\begin{aligned} u_{11} &= u_{13} = u_{14} = 0 \\ u_{12} &= \frac{L_2}{A_3}\Lambda_{1b} \\ w_{01} &= w_{02} = w_{04} = 0 \\ w_{03} &= -\frac{L_2}{2A_3}\Lambda_{1b} \end{aligned} \tag{53}$$

and the expressions of the displacement in the x, z direction are:

$$\begin{aligned} u(x,z) &= \frac{L_2}{A_3}\Lambda_{1b}xz \\ w(x) &= -\frac{L_2}{2A_3}\Lambda_{1b}x^2 \end{aligned} \tag{54}$$

From which the curvature $k = -w''$ is:

$$w'' = -\frac{L_2}{A_3}\Lambda_b \tag{55}$$

Also in the bending case this result is consistent with the higher order solution, which contains in its first terms the first order solution.

5. DISCUSSION

The exact solutions obtained in the previous section are discussed in the following in terms of the geometry of the adaptive system. To this end two parameters are chosen in order to describe the arrangement of the actuators pair and of the passive structure; (i) the ratio $T = t_s/t_a$ between the thickness of the structure and the thickness of the actuator; (ii) the ratio $r = L/t_a$ between the length of the cantilever beam and the thickness of the actuators. Various normalization of the variables of interest are introduced, as described for each particular case.

The membrane solution is first considered. In Figs 2–4 the distribution along the thickness of the axial displacement u (normalized with respect to the one obtained for the Euler–Bernoulli case), of the axial strain ε_x (normalized with respect to the induced strain Λ_{1m}) and of the bending stress σ_x (normalized with respect to $E_a/[(1-\nu_a^2)\Lambda_{1m}]$) are illustrated. In these figures, where the distributions of the variables are shown for different values of the axial coordinates of the beam, it is possible to see the effects of the release of the plane section hypothesis. While the distribution of u and ε_x are regular, the distribution of the σ_x has a jump corresponding to the interface between the actuators and the passive structure, as expected. It is also possible to see that, away from the free edge the high order solution corresponds exactly with the Euler–Bernoulli solution.

Figure 5 describes the distribution of the strain evaluated for $z = h/2$ along the axis of the structure for different values of T. In the vicinity of the free edge the axial strain of the

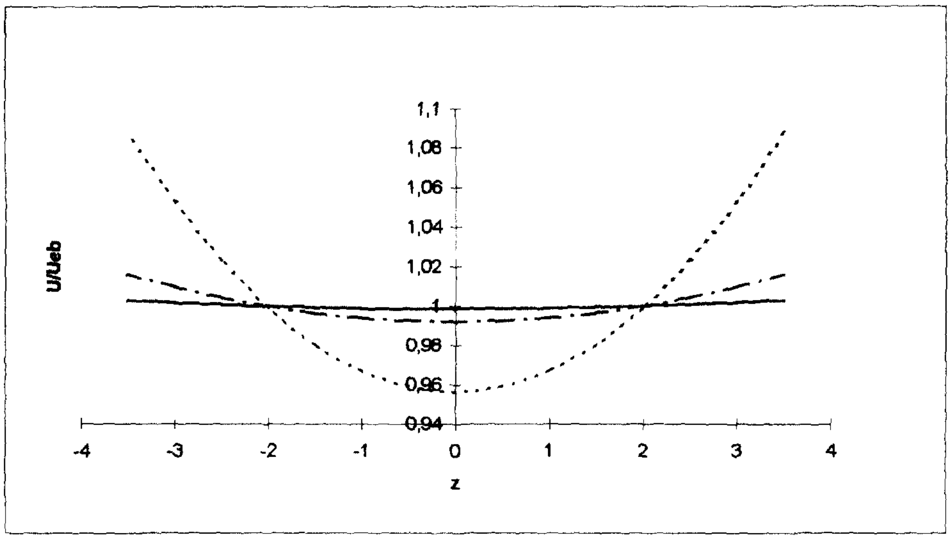

Fig. 2. Axial displacement ($T = 5$; $r = 50$; $x = 0.9L$ solid line; $x = 0.95L$ dashed line; $x = L$ dotted line).

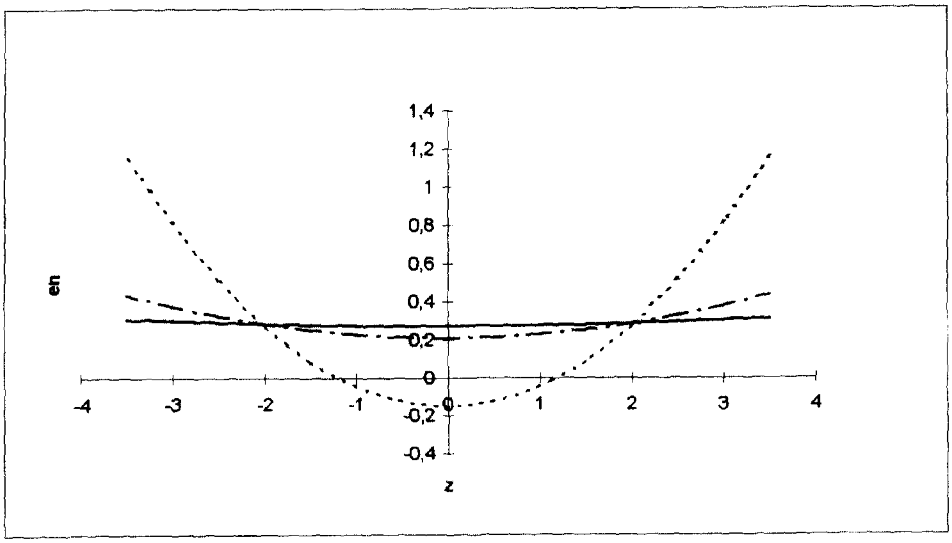

Fig. 3. Axial strain ($T = 5$; $r = 50$; $x = 0.9L$ solid line; $x = 0.95L$ dashed line; $x = L$ dotted line).

structure increases with respect to the Euler–Bernoulli solution, as shown also in Fig. 3. The extension of this effect, which only interest the edge of the beam, depends upon T; the area interested to this effects increases with T.

The pure bending case is considered in Fig. 6 where the tip deflection of the beam for several ratios r is shown as a function of T. From the picture it is possible to note that significant differences with respect to the Euler–Bernoulli model increase with T and assume larger values for low values of r. In Figs 7 and 8 the axial strain (normalized with respect to Λ_{1b}) and the bending stress (normalized with respect to $E_a/[(1-\nu_a^2)\Lambda_{1b}]$) are illustrated for $r = 50$ and $T = 5$ for several values of x. In analogy with the membrane case the removal of plane section hypothesis is evident only in proximity of the edge.

Figure 9 illustrates the induced curvature (normalized with respect to $2\Lambda_{1b}/t_s$) measured at $x = 0.95L$ for $r = 50$, as a function of T. Also in this case the differences between the

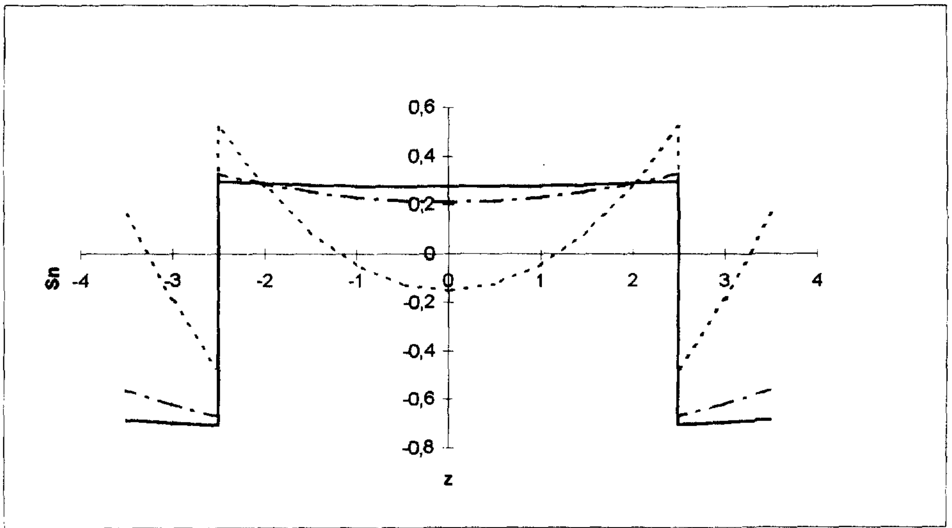

Fig. 4. Axial stress ($T = 5$; $r = 50$; $x = 0.9L$ solid line; $x = 0.95L$ dashed line; $x = L$ dotted line).

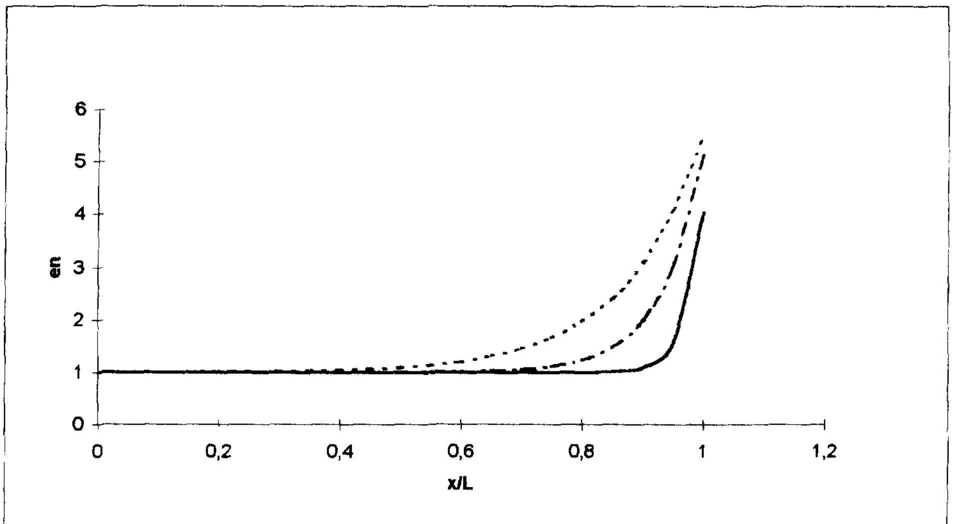

Fig. 5. Axial strain ($z = h/2$; $r = 50$; $T = 5$ solid line; $T = 15$ dashed line; $T = 30$ dotted line).

Euler–Bernoulli solution and the present one are important only as T becomes large. The "Pin-force" results (1) are also shown for comparison.

In Fig. 10 the curvature is normalized with respect to Λ_{1b}. Euler–Bernoulli (solid line) and high order solution show a relevant difference only as T increases as better illustrated in Fig. 11 where the curvature is plotted starting from $T = 5$. For low values of T the higher order and the Euler–Bernoulli models give the same results.

From Fig. 12, in which the curvature is normalized with respect to the corresponding Euler–Bernoulli solution, the extension of the edge effect for different values of T can be examined. The larger is T the more extended is the portion of the beam interested by the effect. As r and L are fixed (in this case $L = 50$, $r = 50$) the effect of the increase of T can be interpreted as a shear deformation effect, represented by higher order terms of the power expansion of displacement components in terms of z.

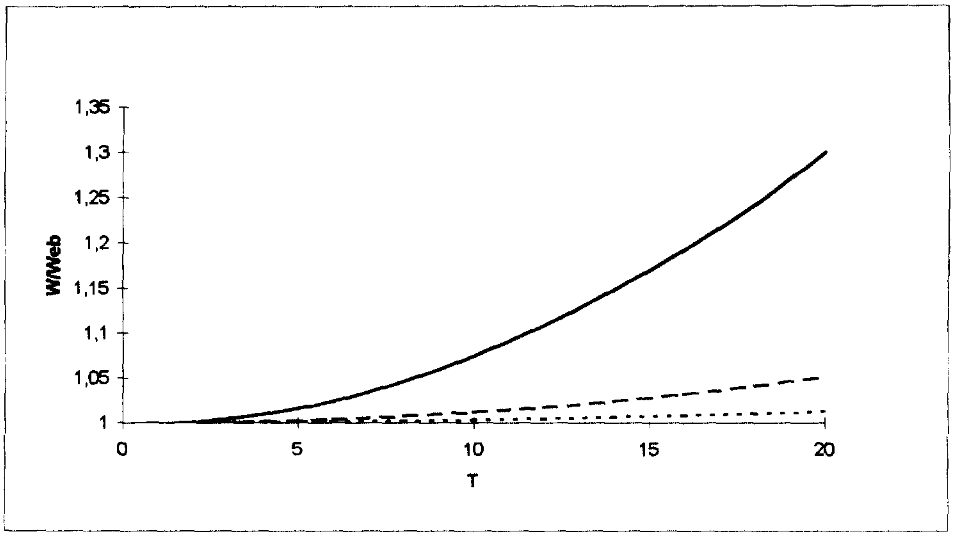

Fig. 6. Normalized deflection ($x = L$; $r = 10$ solid line; $r = 25$ dashed line; $r = 30$ dotted line).

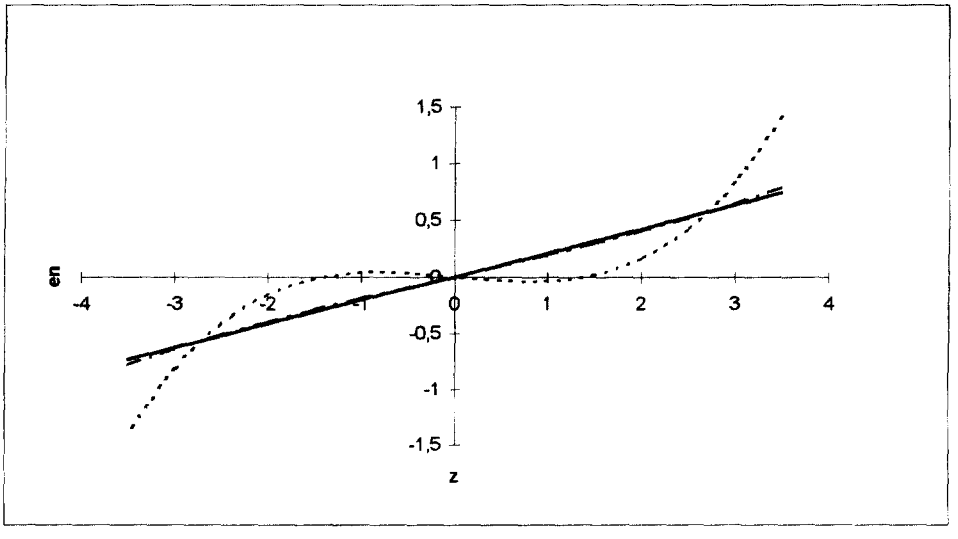

Fig. 7. Axial strain ($T = 5$; $r = 50$; $x = 0.9L$ solid line; $x = 0.95L$ dashed line; $x = L$ dotted line).

6. CONCLUSIONS

The proposed model, although very simple from the point of view of the formulation, has permitted one to evaluate the importance of higher order effects in the modeling of the interaction between induced strain actuators and a simple beam structure. The closed form solution obtained for the membrane and bending cases has allowed one to identify the main geometrical and material parameters of the problem and their role in the response of the system. By examining the results obtained with this model, the range of applicability of the classical Euler–Bernoulli model are better assessed. Moreover the model permits one to predict the extension of the area interested by the edge effect. The mechanical interaction between surface bonded induced strain actuators and passive structure is evaluated in more detail than with classical beam model with possible advantages for the design process.

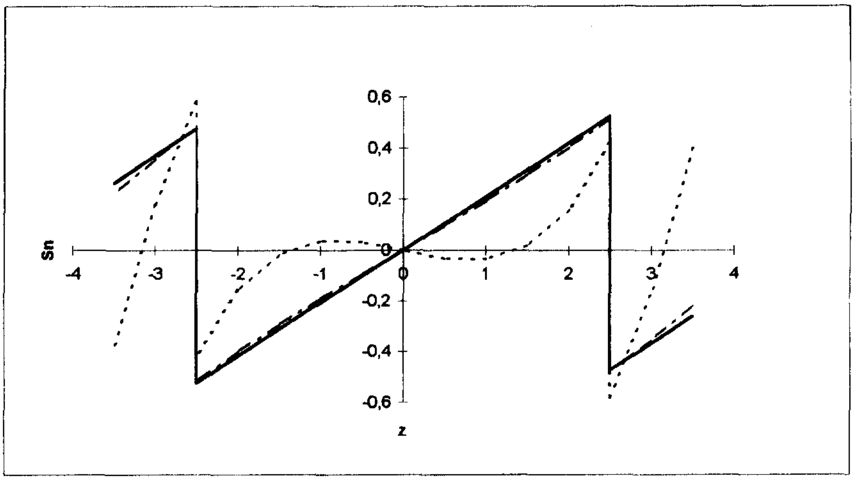

Fig. 8. Axial stress ($T = 5$; $r = 50$; $x = 0.9L$ solid line; $x = 0.95L$ dashed line; $x = L$ dotted line).

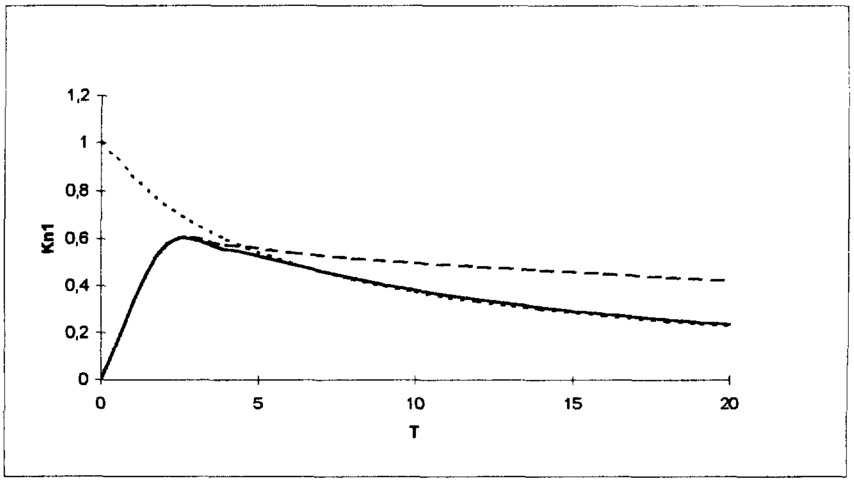

Fig. 9. Normalized curvature $Kt_s/2\Lambda_{1b}$ ($r = 50$; $x = 0.95L$; pin-force: dotted line; Euler–Bernoulli: solid line; higher order: dashed line).

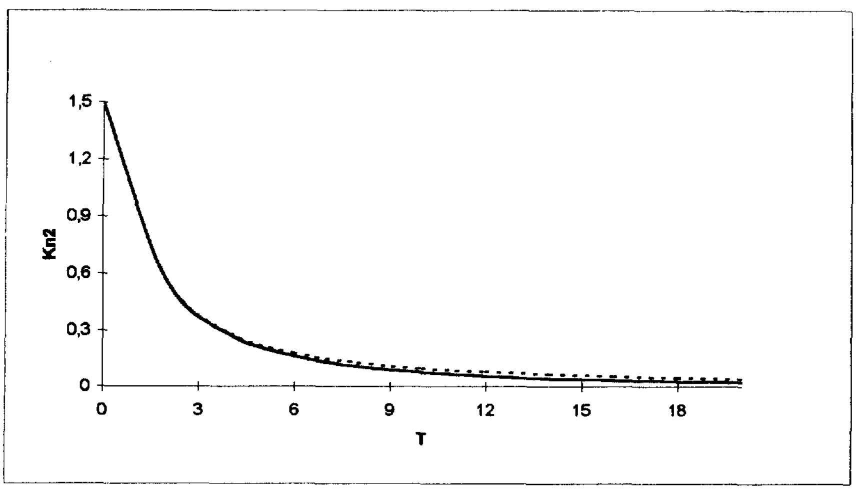

Fig. 10. Normalized curvature K/Λ_{1b} ($r = 50$; $x = 0.95L$; Euler–Bernoulli: solid line; higher order: dotted line).

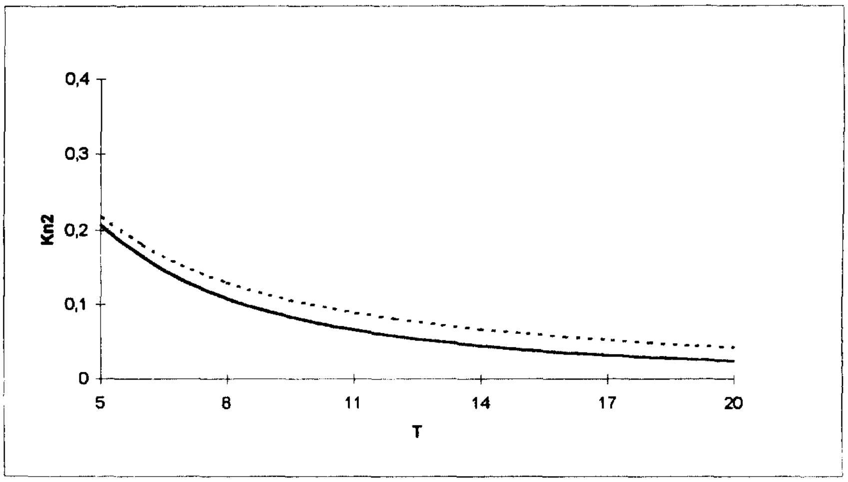

Fig. 11. Normalized curvature K/Λ_{1b} ($r = 50$; $x = 0.95L$; Euler–Bernoulli: solid line; higher order: dotted line).

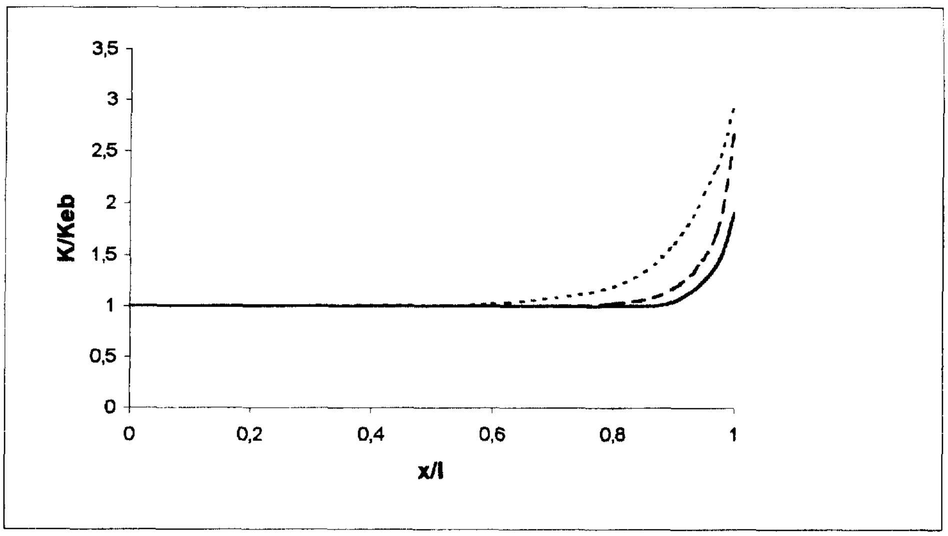

Fig. 12. Normalized curvature K/K_{eb} ($r = 50$; $T = 5$ solid line; $T = 15$ dashed line; $T = 30$ dotted line).

Acknowledgement—The financial support of the research project "Controllo delle vibrazioni di semplici strutture intelligenti" (MURST- Italian Ministry for University and Research—Rif. 410/1995) is gratefully acknowledged.

REFERENCES

Barboni, R., Gaudenzi, P. and Strambi, G. (1995) On the modeling of actuator–structure interaction in intelligent structures. *Intelligent Materials and Systems, Techna*, pp. 291–298.

Bisegna, P. and Maceri, F. (1996) An exact three-dimensional solution for simply supported rectangular piezoelectric plates. *ASME Journal of Applied Mechanics* **63**, 628–638.

Crawley, E. F. (1994) Intelligent structures for aerospace: a technology overview and assessment. *AIAA Journal* **32**(8), 1689–1699.

Crawley, E. F. and de Louis, J. (1987) Use of piezoelectric actuators as element of intelligent structures. *AIAA Journal* **25**(10), 1373–1385.

Denoyer, K. K. and Kwak, M. K. (1996) Dynamics modeling and vibration suppression of a slewing structure utilizing piezoelectric sensors and actuators. *Journal of Sound and Vibration* **189**(1), 13–31.

Gaudenzi, P. (1997) On the electromechanical response of active composite materials with piezoelectric inclusion. *Computers and Structures* **65**(2), 157–168.

Gaudenzi, P. and Bathe, K. J. (1995) An iterative finite element procedure for the analysis of piezoelectric continua. *Journal of Intelligent Material System and Structures* **6**, 266–273.

Gaudenzi, P., Barboni, R. and Mannini, A. (1995) A finite element evaluation of single-layer and multi-layer theories for the analysis of laminated plates. *Composite Structures* **30**, 427–440.

Gaudenzi, P., Barboni, R., Carbonaro, R. and Accettella, S. (1997) Direct position and velocity feed-back control on an active beam with PZT sensors and actuators. *International Forum on Aeroelasticity and Structural Dynamics*, 17–20 June 1997, Vol. II, pp. 285–292 Rome, CEAS.

Kim, J., Varadan, V. V. and Varadan, V. K. (1997) Finite element modeling of structures including piezoelectric active devices. *International Journal for Numerical Methods in Engineering* **40**, 817–832.

Ray, M. C., Rao, K. M. and Samanta, B. (1993) Exact solution for static analysis of intelligent structures under cylindrical bending. *Computers and Structures* **47**(6), 1031–1042.

Reddy, J. N. (1987) A generalization of two-dimensional theories of laminated composite plates. *Common Appl. Numer. Meth.* **3**, 173–181.

Saravanos, D. A. and Heyliger, P. R. (1995) Coupled layerwise analysis of composite beams with embedded piezoelectric sensors and actuators. *Journal of Intelligent Material System and Structures* **6**, 350–363.

Saravanos, D. A., Heyliger, P. R. and Hopkins, D. A. (1997) Layerwise mechanics and finite element for the dynamics analysis of piezoelectric composite plates. *International Journal of Solids and Structures* **34**, 359–378.